Proposition 1. *Let $f : A \to B$ be a function. Let $W, X \subset A$ and $Y, Z \subset B$. Then:*

(1) $f(W \cap X) \subset f(W) \cap f(X)$
(2) $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$
(3) $X \subset f^{-1}(f(X))$

Proof.

(1) Let $b \in f(W \cap X)$, then $\exists a \in W \cap X$ such that $f(a) = b$. Since $a \in W \cap X$, we know $a \in W$ and $a \in X$. Thus $f(a) = b \in f(W)$ and $f(a) = b \in f(X)$, thus $b \in f(W) \cap f(X)$.
$\therefore f(W \cap X) \subset f(W) \cap f(X)$.
To see that the reverse containment is not true, observe the following counter example: let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$, $W = [-2, -1]$, and $X = [1, 2]$. Then $W \cap X = \varnothing$ so that $f(W \cap X) = \varnothing$. On the other hand, $f(W) = [1, 4]$ and $f(X) = [1, 4]$, so we see that $f(W) \cap f(X) = [1, 4]$. Therefore, we have
$$f(W) \cap f(X) = [1, 4] \not\subset f(W \cap X) = \varnothing,$$
giving us a counter example to containment in the other direction.

(2) Let $a \in f^{-1}(Y \cup Z)$. Then $f(a) \in Y \cup Z$. So $f(a) \in Y$ or $f(a) \in Z$. It follows that $a \in f^{-1}(Y)$ or $a \in f^{-1}(Z)$ which gives $a \in f^{-1}(Y) \cup f^{-1}(Z)$.
$\therefore f^{-1}(Y \cup Z) \subset f^{-1}(Y) \cup f^{-1}(Z)$
Notice that all of the above steps are reversible, which will give us that
$$f^{-1}(Y) \cup f^{-1}(Z) \subset f^{-1}(Y \cup Z)$$
$\therefore f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.

(3) Let $x \in X$. Then $f(x) \in f(X)$. By definition of preimage, since $f(x) \in f(X)$, this means that $x \in f^{-1}(f(X))$. Therefore $X \subset f(f^{-1}(X))$.
To see that the containment does not go the other way, we can refer to the example from class. Here is another counterexample:
Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$ and let $X = [0, 1]$. Then $f(X) = [0, 1]$ and $f^{-1}([0, 1]) = [-1, 1]$, which shows that $f^{-1}(f(X)) \not\subset X$ in general.

□